Generalization of Artin's Conjecture

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DEFINITIONS

1. primitive root mod p: a prime number p is a primitive root modulo a if **the smallest natural number k such that a^k ≡1 mod p is p-1**.

INTRODUCTION

Artin's Conjecture on primitive roots states that an integer a that is neither a **square number** nor **-1** is a primitive root mod infinitely many primes p. Furthermore if a is **not** a **perfect power** and a≠ 1 (mod 4), then the density of primes p such that a is a primitive root modulo p is independent of a and equals **Artin's Constant**.

Goldmakher and **Martin** seek to generalize this behaviour with their conjecture which considered additionally **how far** a is a primitive root modulo a prime.

Fig 1. Emil Artin

OBJECTIVE

Reinforce the **Goldmakher-Martin conjecture** by testing it against **empirical data** and demonstrating a high statistical correlation.

METHODOLOGY

Numerical Approximation techniques were used to provide **evidence** for the Goldmakher-Martin conjecture and **obtain predicted values** for our empirical data. This included the Integral Test and logarithm tricks to find the values and obtain **error bounds** for them.

Python, in particular **Sage**, was used to **gather empirical data** for the hypothesis. Visualization and Analysis is performed using R.

CHALLENGES

1. One of the biggest challenges was **understanding the research** and how the Goldmakher-Martin Conjecture worked
2. Another challenge was **optimizing the data collection code**. While we have attempted to implement our own algorithms for steps such as factorization and take large exponents, the sage library proved to be more optimized.
3. Finding the appropriate analysis technique to visually analyze the histograms created, was one of the challenges we faced, hence there's more than one way to approach this, but the method we used proved to be the easiest to understand, and analyze.

ANALYSIS

The analysis conducted in our research included both visual and analytical methods. For the visual analysis, we plotted histograms for all the exponents from Goldmakher-Martin conjecture, for random values of a, showing the frequency for various function involving the order of a modulo primes, for example the difference between the order of a modulo p and p-1 for different primes p, this histogram was plotted on top of the histogram generated from theoretical values to prove the correlation of the results. For the analytical methods, we used 2 statistical methods which is the Mann-Whitney U test, and the Kolmogorov-smirnoff test, these tests were used to prove there is a strong correlation between the shapes of the histograms.

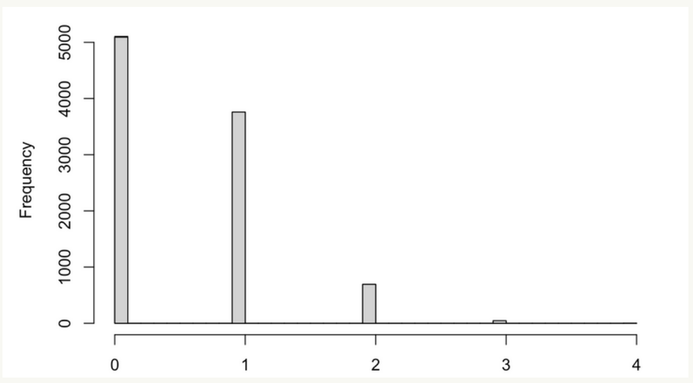

Fig 2. Sample of the overlapping histogram

FUTURE DIRECTIONS

1. We had collected data for integers a from -400 to 400 for all primes up to a million. A step that can be taken to improve the quality of the results is to **collect more data**.
2. While we had developed a method for obtain error bounds on our predicted values, we could not perform the computations in time. So, **finishing the computation for the error bounds** is a logical future step that can be taken.
3. For the data analysis, we have used 3 approaches to analyze the data collected, and hence arrive at our conclusion. **Introducing other techinques to analyze the data** would be useful as a future step.

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