# PACIFIC INSTITUTE FOR THE MATHEMATICAL SCIENCES VIRTUAL EXPERIMENTAL MATHEMATICS LAB (PIMS VXML) FINAL REPORT: CHIP FIRING GAMES: APPLICATIONS OF THE SMITH NORMAL FORMS OF COMBINATORIAL MATRICES

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## 1. Introduction

1.1. **The initial problem.** The initial problem was to use Laplacian matrices to study chip-firing games. Laplacian matrices can be explored using Smith Normal Forms. The Smith Normal Form (SNF) of a matrix is a powerful invariant that may help distinguish the underlying structures of Laplacian matrices.

There are several versions of chip-firing games. One of the many games can be found here [3].

1.2. **New directions.** The discussion began by exploring chip-firing games and their Laplacian matrices. From here, the focus of the project was directed towards exploring Smith Normal Forms. For this project, we primarily concentrated on what Smith Normal Forms are, how they are computed, and computing Smith Normal Forms of different combinatorial matrices.

To facilitate our research, we studied and discussed two papers. These papers provided us with a deep understanding of Smith Normal Forms and introduced us to several new concepts. The textbook by Wallis [4] provided some background information on Hadamard matrices, which are a type of combinatorial matrices we investigated. In addition, we learned to use Sage and LaTeX for ease of calculations and collaboration.

## 2. Progress

2.1. **Computational.** Computations for Smith Normal Forms were mostly done by hand, and Sage was occasionally used for the ease of calculations. Smith Normal Forms are defined below.

**Smith Normal Forms (SNF):** The Smith Normal Form of a matrix is the matrix diagonal that can be obtained by taking a $m \times n$ integral matrix $A$, and multiplying it on the left by matrix $P$, and on right by matrix $Q$ (where both $P$ and $Q$ are square, invertible, integral matrices, and their inverses are also integral). SNF of a matrix looks like the following:

$$\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $d_1|d_2|d_3 \cdots |d_n$. These $d_i$'s are called the invariant factors.

To compute the Smith Normal Form, one can identify matrices $P$ and $Q$, that $A$ can be multiplied by. Alternatively, the algorithm shown below can be used to determine the invariant factors of $A$:

$d_1$=$gcd$ of all $|x|$ minors of the original matrix.
$d_2 d_1 = gcd$ of all $2 \times 2$ minors of the original matrix.
$\vdots$
$d_i d_{i-1} \cdots d_1$=$gcd$ of all $i \times i$ minors of the original matrix.

Below is a worked example for a better understanding of the algorithm:

Let $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ be the given matrix. Using the algorithm described above the invariant factors are determined to be:

$$d_1 = gcd(1,0) = 1,\ d_2 = \frac{gcd(1,-1,0)}{d_1} = 1,\ d_3 = \frac{gcd(1)}{d_2 d_1} = 1$$

2.2. **Theoretical.** After gaining an understanding of how Smith Normal Forms work, and before studying Smith Normal Forms of combinatorial matrices, we deduced the Smith Normal Forms of Laplacian matrices of cyclic graphs with $n$ vertices. It turned out to be of the following:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Following this, we looked at a type of combinatorial matrix known as Hadamard matrices, its different sub-classes, and their respective Smith Normal Forms. Two papers were analyzed, and the key results are described below.

**First Paper [1]:**

The first paper was on the Smith Normal Form of the Skew-Hadamard matrices, written by T.S. Michael and W. D. Wallis. This paper provided us with a deep understanding of Smith Normal forms and introduced us to several new concepts like Skew-Hadamard matrices, Skew-Hadamard Design and p-ranks.

The definitions are briefly discussed below:

**Hadamard matrices:** A Hadamard matrix of order $n$ is a $(1, -1)$-matrix $H$ that satisfies $H^\top H = nI$.

**Skew-Hadamard matrices:** A Hadamard matrix is the matrix that satisfies the equation $H + H^\top = 2I$.

**P-rank:** The p-rank of a matrix A is defined as:

$$rank_p(A) = max\{i : p \text{ does not divide } a_i\}$$

where $a_1, a_2, \cdots a_r$ are invariant factors.

Using these key definitions and some well-known results author concluded that the Smith Normal Form of the Skew-Hadamard matrix is:

diag[1, 2,..., 2, 2m,..., 2m, 4m]

where 2 and $2m$ occurs exactly $2m - 1$ times.

**Second Paper [2]:**
The second paper written by Morris Newman discusses the Smith Normal Forms of Hadamard matrices of order $4m$, where $m$ is a square-free number. The proof is as follows:

Given $A$ is an $n \times n$ non-singular integral matrix, then

$$SNF(A^{\top}) = SNF(A)$$

Since $H$ is a Hadamard matrix of size $n \times n$, where $n = 4m$, and $H$ satisfies the property $HH^{\top} = nI$, then

$$H^{\top} = nH^{-1}$$

and therefore

$$SNF(H) = SNF(nH^{-1})$$

Given that the Smith Normal Form of a Hadamard matrix $H$ is $SNF(H) = diag(h_1, h_2, ..., h_n)$, the $k^{th}$ invariant factor of $H$ is

$$h_k = n/h_{n-k+1}, \quad 1 \leq k \leq n$$

Therefore, we have

$$SNF(H) = diag(h_1, h_2, ..., h_{2m}, n/h_{2m}, n/h_{2m-1}, ..., n/h_1)$$

where

$$h_k | h_k + 1, 1 \leq k \leq 2m - 1, h_{2m}^2 | 4m$$

Now, given $m$ is square-free, then this implies that $h_{2m}^2 | 4m$, and thus $h_{2m}$ can either be 1 or 2. Taking rank of $H$ modulo 2, we get 1, since $H \equiv J \mod 2$, where $J$ is a $n \times n$ matrix whose each entry is 1. Therefore, the invariant factors of a $n \times n$ Hadarmard matrix, where $m$ is square-free is

diag[1, 2,..., 2, 2m,..., 2m, 4m]

where 2 and $2m$ occurs exactly $2m - 1$ times.

## 3. Future directions

Throughout our research, we stumbled upon several interesting questions. It turned out that the Smith Normal Forms of Skew-Hadamard matrices and Hadamard matrices of order $4m$ (where $m$ is square-free) are the same.

Thus, it might be interesting to find other classes of matrices that share the exact same Smith Normal Form. One can also explore other similarities between Skew-Hadamard matrices and square-free Hadamard matrices. Additionally, it would be interesting to see what difference $m$ being square free makes in terms of the Smith Normal Form of Hadamard matrices.

Since the two matrix sub-classes discussed share the same Smith Normal Form, and given that Smith Normal Forms are a commonly used to distinguish between different types of matrices, it would be interesting if one can deduce other methods to distinguish between square-free and Skew-Hadamard matrices.

## References

[1] Michael, & Wallis, W. D. (1998). Skew-Hadamard Matrices and the Smith Normal Form. Designs, Codes, and Cryptography, 13(2), 173–. https://doi.org/10.1023/A:1008230429804

[2] Newman. (1971). Invariant factors of combinatorial matrices. Israel Journal of Mathematics, 10(1), 126–130. https://doi.org/10.1007/BF02771523

[3] The dollar game. (n.d.). Retrieved April 28, 2023, from https://thedollargame.io/game/level/100/100/1

[4] Wallis. (2016). Introduction to Combinatorial Designs (Vol. 38). CRC Press.