CHIP FIRING GAMES: APPLICATIONS OF THE SMITH NORMAL FORMS OF COMBINATORIAL MATRICES

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MOTIVATION

- Introduced by Bjorner et al., the chip-firing game involves a graph where each vertex (v) has a certain number of chips
- With each step, vertex v is "fired", and a single chip goes from the vertex to its adjacent vertices
- The Laplacian matrix appears in the context of the chip-firing game, and its structure can be studied via Smith Normal Forms (SNFs)
- From the Laplacian matrix and chip-firing games, the focus of this project was directed towards investigating SNFs and looking at SNFs of a few combinatorial matrices

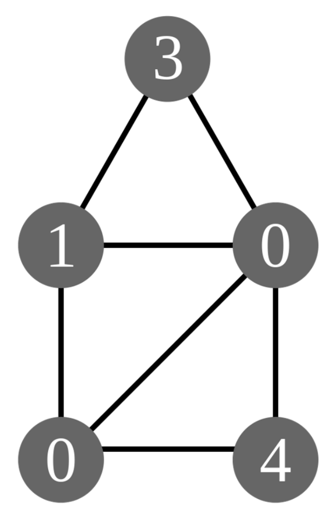

Figure 1. An example of a chip-firing game graph

INTRODUCTION TO SNFs

- Suppose A is $m \times n$ integral matrix
- One can obtain SNF for A by multiplying it by matrix P and Q as shown:

$$SNF(A) = P \cdot A \cdot Q = \begin{pmatrix} \alpha_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \alpha_3 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \alpha_r & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

- Where P and Q are square, invertible, and integral matrices, and their inverses (P^{-1} and Q^{-1}) are also integral
- The non-zero entries in the diagonal of SNF are referred to as invariant factors
- Invariant factors must satisfy the condition $\alpha_i | \alpha_{i+1}$ for all $1 \le i < r$, where r is the rank of SNF(A)
- SNF of every matrix exists and is unique
- To determine SNF of A, one can identify the elementary row and column operations matrices P and Q would apply onto A
- Alternatively, the following algorithm can be used to calculate the invariant factors:

$$\alpha_1 = \gcd of\ all\ |x|\ minors\ of\ A$$
$$\alpha_2 \cdot \alpha_1 = \gcd of\ all\ 2 \times 2\ minors\ of\ A$$
$$\vdots$$
$$\alpha_i \cdot \alpha_{i-1} \cdot \dots \cdot \alpha_1 = \gcd of\ all\ i \times i\ minors\ of\ A$$

HADAMARD MATRICES

- Hadamard matrices are a type of combinatorial matrix, that possess the following properties:
  - Are a square matrix of order n
  - Consists of only +1 and -1 as its entries
  - $H^T \cdot H = nI$ (where I is the identity matrix)

$$\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

Figure 2. An example of a Hadamard matrix

SNFs OF HADAMARD MATRICES

- 2 subclasses of Hadamard matrices and their invariant factors were investigated
- Skew-Hadamard matrices [1]:
  - Satisfy the condition $H + H^T = 2I$
  - Using known results of SNFs and p-ranks, the invariants factors were determined to be:

$$1\ (once), 2\ (2m-1\ times), 2m\ (2m-1\ times), 4m\ (once)$$

- Hadamard matrices of order v [2]:
  - Satisfy the condition $v = 4m$, where m is a square-free number
  - Using known properties of SNFs and rank of Hadamard matrices modulo 2, the invariant factors were determined to be:

$$1\ (once), 2\ (2m-1\ times), 2m\ (2m-1\ times), 4m\ (once)$$

- Hence, both of the subclasses of Hadamard matrices studied were found to have the same invariant factors

FUTURE DIRECTIONS

- For the future, it may be interesting to investigate the following:
  - Determine other classes of matrices that share the same SNF
  - Determine whether there are other similarities between Skew-Hadamard Matrices and Hadamard matrices of order $4m$ (where m is square-free)

REFERENCES

[1] Michael, T. S., & Wallis, W. D. (1998). Skew-Hadamard matrices and the Smith normal form. *Designs, Codes and Cryptography*, *13*, 173-176.
[2] Newman, M. (1971). Invariant factors of combinatorial matrices. *Israel Journal of Mathematics*, *10*, 126-130.