

TOPOLOGICAL DATA ANALYSIS OF COMMUTING PAIRS OF $SU(2)$

ZHIKAI CHEN, PRATHAM LALWANI, DINAMO DJOUNVOUNA, DEREK
KREPSKI, STEVEN RAYAN

1. INTRODUCTION

Topology is a branch of mathematics that studies the properties of objects that remain unchanged when the object is stretched, bent, or twisted. In recent years, topological methods have become increasingly popular in data analysis, as they provide a powerful tool for extracting meaningful information from complex datasets.

One key concept in topology is **homology**, which measures the number of holes in an object of a given dimension. Homology provides a way to distinguish different shapes and structures and has been used to analyze a wide range of datasets, from images and videos to gene expression data and networks.

Topological Data Analysis (TDA) is a field that uses algebraic topology to analyze complex data sets. One of the main techniques in TDA is **Persistence Homology**, which assigns a measure of persistence to topological features in a data set. In this project, we apply Persistence Homology to a data set of points of commuting pairs of matrices in the Lie group $SU(2)$. We used the Ripser software to compute the persistence homology of our data set. **Ripser** is a C++ program that computes persistence homology using the **Vietoris-Rips filtration**.

Our method uses a combination of topological and algebraic techniques to construct a simplicial complex from a point cloud and then computes the persistent homology of this complex to identify the topological features of the space.

Our results show that there is a connected component in the space of commuting pairs of matrices in $SU(2)$, but no higher-dimensional features such as loops. We compare our results with previous work in the field and identify several limitations and challenges that need to be addressed in future research.

2. THE INITIAL PROBLEM

The study of the topological properties of the set of commuting pairs of $SU(2)$ is highly relevant due to its applications in various fields such as physics and geometry. The set of commuting pairs of $SU(2)$ arises in the study of the spectral theory of quantum mechanical systems with two degrees of freedom, which has significant implications in the field of quantum mechanics. Moreover, understanding the topological properties of this set can provide insights into the geometric structures of the space in which the $SU(2)$ group acts, which has applications in the study of geometry and topology. Therefore, investigating the topological properties of the set of commuting pairs of $SU(2)$ is crucial in advancing our understanding of various complex systems in physics and mathematics.

This project aims to investigate the topological properties of the set of commuting pairs in $SU(2)$ using TDA. The set of commuting pairs arises in the study of the spectral theory of quantum mechanical systems with two degrees of freedom and has significant implications in the field of quantum mechanics. The study of the topological properties of this set can also provide insights into the geometric structures of the space in which the $SU(2)$ group acts, which has applications in the study of geometry and topology.

The project methodology involved the following steps: data collection, persistent homology, analysis of persistent homology, and computation of persistent homology barcodes. The data set used in this project was obtained from the set of commuting pairs in $SU(2)$, and a point was obtained in the four-dimensional Euclidean space \mathbb{R}^8 for each pair of commuting elements. The persistent homology analysis used the Rips filtration to construct a simplicial complex from the point cloud and extract topological features. The persistence diagram and barcode were computed to visualize and characterize these features.

3. NEW DIRECTIONS

As our research team worked on the problem of studying the topological properties of the set of commuting pairs of $SU(2)$, we explored several different directions in order to gain a better understanding of the problem and to identify the most effective approaches for analyzing the data.

At the beginning of the project, we started by researching the relevant literature and discussing various techniques for analyzing point clouds. We quickly settled on the use of TDA and decided to use persistent homology to extract topological features from the data set. We

also decided to employ the Ripser filtration as our primary method for constructing the simplicial complex from the point cloud.

Following the selection of our methodology, our research team embarked on the data collection phase, during which we acquired a dataset consisting of 1000 points from the set of commuting pairs of $SU(2)$. These points were then used to construct a point cloud for analysis, which we generated using our custom algorithms - the code of which can be found in this report. Our initial focus during the analysis stage was to compute the persistence diagram for the dataset. By doing so, we were able to visualize the emergence and disappearance of topological features as the distance parameter was progressively increased.

After computing the persistence diagram, we analyzed the zeroth and first homology groups to determine the number of connected components and loops in the data set. We also computed persistent homology barcodes to obtain a more detailed characterization of the topological features present in the data set. These analyses allowed us to identify the most significant topological features of the data set, namely a single connected component and no topologically non-trivial loops.

As we continued our analysis, we encountered several limitations and challenges that we had to overcome. For example, we realized that our current approach was limited to computing features of dimensions 0 and 1, while there may exist higher dimensional features up to dimension 4 in the data. We also discovered that the computational cost associated with creating simplices from point clouds in higher dimensions was prohibitively expensive, posing a significant challenge for analyzing point clouds with a large number of dimensions.

To address these limitations and challenges, we explored alternative techniques for analyzing point clouds, such as the use of machine learning algorithms and other TDA methods. We also considered the possibility of increasing the sample size to obtain more accurate results for higher dimensional features.

Throughout our work on this project, we learned a great deal about the topological properties of the set of commuting pairs of $SU(2)$ and the challenges associated with analyzing point clouds. We also gained a deeper understanding of TDA and its applications in various fields, particularly in physics and geometry. Overall, our research team was able to make significant progress in studying the topological properties of the set of commuting pairs of $SU(2)$ and identified several avenues for future research.

4. PROGRESS

Through our research, we were able to answer some of the questions we started with and those that emerged during our work. We were able to determine the topological properties of the set of commuting pairs of $SU(2)$ using TDA. Specifically, we found that the set has one connected component and no loops, which is consistent with the results obtained by Adem and Cohen. We were also able to determine the limitations and challenges associated with our approach, which can help guide future research in this area.

However, our approach was limited to computing features of dimensions 0 and 1, while higher dimensional features up to dimension 4 may exist in the data. This limitation means that we may have an incomplete understanding of the underlying topological structure. Additionally, our method of sampling point cloud may introduce bias, which may impact the accuracy of our results.

Throughout our research, we pursued different directions, including exploring various TDA techniques and experimenting with different sample sizes. We learned that the computational cost associated with creating simplices from point clouds in higher dimensions is prohibitively expensive, which poses a significant challenge for analyzing point clouds with a large number of dimensions. We also learned that obtaining additional sample points could potentially address some of the limitations associated with our approach.

Overall, our research provides valuable insights into the topological properties of the set of commuting pairs of $SU(2)$ and highlights the challenges associated with using TDA to analyze high-dimensional data.

5. COMPUTATIONAL

5.1. Sample point to get a point cloud. The research team generated a point cloud for analysis by collecting a set of 1000 points from the set of commuting pairs of $SU(2)$ using their own developed algorithms. They utilized Python code to generate random elements of $SU(2)$ and checked if their product commutes. The generated points were then plotted in 3D. Here is the Python code for generating a random point cloud in $SU(2)$ of size N :

```
# Helper function to generate a random element of SU(n)
def random_su(n):
    A = np.random.rand(n, n) + 1j * np.random.rand(n, n)
    Q, R = np.linalg.qr(A)
    D = np.diag(np.exp(1j * np.random.rand(n) * 2 * np.pi))
```

```

return np.dot(np.dot(Q, D), np.linalg.inv(Q))

# Generate a random point cloud in SU(n) of size N

N = 1000
n = 3
X = []
A = [random_su(n) for i in range(1000)]
B= [random_su(n) for j in range(1000)]
while len(X)<N:
for i in range(1000):
for j in range(1000):
if np.allclose(np.dot(A[i], B[j]), np.dot(B[j], A[i])):
X.append(A[i])
X.append(B[j])
A = [random_su(n) for i in range(1000)]
B = [random_su(n) for j in range(1000)]
# while len(X) < N:
#     B = random_su(n)
#     if np.allclose(np.dot(A, B), np.dot(B, A)):
#         X.append(B)

%% md
|a|^2 + |b|^2 = 1

|\alpha|^2 + |\beta|^2 = 1

\bar{\beta} = \bar{b}\beta

```

5.2. **Barcode.** To compute the barcode of a point cloud that has been sampled from X , we need to determine the persistence homology of the associated filtration. This filtration is characterized by a sequence of nested simplicial complexes, where the k th complex represents the union of all simplices that are formed by $k + 1$ or fewer points in the point cloud. The persistence homology of this filtration gives us the barcode, which encodes the lifetimes of the topological features such as connected components, loops, voids in the point cloud as they appear and disappear.

To compute the barcode, we can use a TDA library such as ripser or Gudhi in Python. Here's an example that utilizes ripser:

```
# Compute the persistence homology
points = np.array([np.linalg.eig(x)[0] for x in X])
rips_complex = gudhi.RipsComplex(points=points, max_edge_length=1)
simplex_tree = rips_complex.create_simplex_tree()
diag = simplex_tree.persistence()
# Print the barcodes for each homological dimension
for dim in range(3):
    print(f"Dimension {dim}:")
    gudhi.plot_persistence_barcode(diag)
```

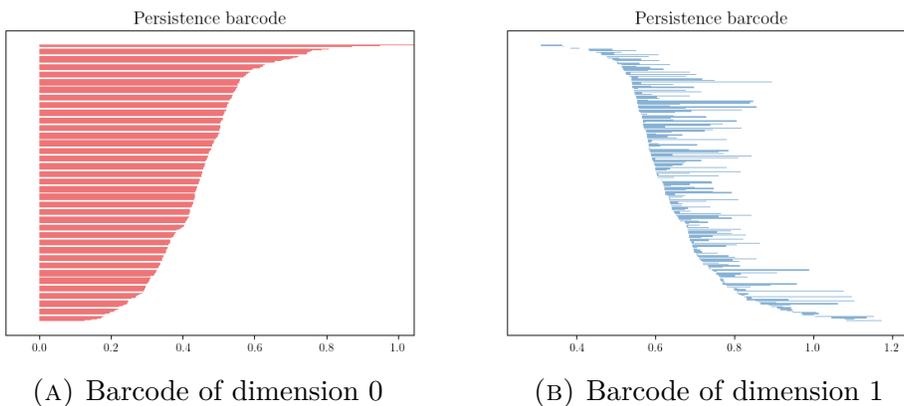


FIGURE 1. Barcode of dimension 0 and 1 of the data sampled from the space of commuting pairs in $SU(2)$

The left figure's barcode indicates that the point cloud has a single 0-dimensional feature, specifically a connected component that persists until the maximum edge length of 10000. This feature is represented by the topmost line on the barcode. Conversely, the right figure's barcode does not contain any lines that are long enough to be considered significant. Consequently, we can infer that there are no topologically non-trivial loops in our point cloud.

Our findings align with the results derived by Adem and Cohen [2], which are expressed as:

$$H^i(\text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, SU(2)), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \end{cases}$$

This equation indicates that the space of commuting pairs of $SU(2)$ has a connected component and no loops, which corresponds to our TDA results.

6. THEORETICAL

7. THE SET OF COMMUTING ELEMENTS IN $SU(2)$

In this section, we will study the set of commuting elements in the group $SU(2)$. This set consists of pairs of matrices (A, B) in $SU(2)$ that commute, that is, $AB = BA$ for all $B \in SU(2)$. Our goal is to describe this set in terms of equations with real coefficients, which will allow us to view it as the solution set of a system of (non-linear) equations.

To do this, we will start by computing the product of two matrices in $SU(2)$ and then use the condition that $AB = BA$ to derive a system of equations that must be satisfied by any pair of matrices in $SU(2)$ that commute. By solving this system of equations, we will obtain a description of the set of commuting elements in $SU(2)$ in terms of real coefficients.

Definition 7.1. *The group $SU(2)$ is defined by*

$$SU(2) = \left\{ M = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1 \right\}$$

Consider the

$$X = \{(A, B) \in SU(2) \times SU(2) : AB = BA\}$$

Let $A = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}$ such that $(A, B) \in X$. Then $|a|^2 + |b|^2 = 1$, $|\alpha|^2 + |\beta|^2 = 1$, and $AB = BA$. This means First, let's compute AB :

$$AB = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} = \begin{bmatrix} a\alpha - \bar{b}\beta & -a\bar{\beta} - \bar{b}\bar{\alpha} \\ b\alpha + \bar{a}\beta & -b\bar{\beta} + \bar{a}\bar{\alpha} \end{bmatrix}$$

Now, let's compute BA :

$$BA = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} = \begin{bmatrix} \alpha a - \bar{\beta}b & -\alpha\bar{b} - \bar{\beta}\bar{a} \\ \beta a + \bar{\alpha}b & -\beta\bar{b} + \bar{\alpha}\bar{a} \end{bmatrix}$$

Since $AB = BA$, we obtain we have:

$$\begin{aligned} a\alpha - \bar{b}\beta &= \alpha a - \bar{\beta}b \\ -a\bar{\beta} - \bar{b}\bar{\alpha} &= -\alpha\bar{b} - \bar{\beta}\bar{a} \\ b\alpha + \bar{a}\beta &= \beta a + \bar{\alpha}b \\ -b\bar{\beta} + \bar{a}\bar{\alpha} &= -\beta\bar{b} + \bar{\alpha}\bar{a} \end{aligned}$$

We observe that the first equation has a complex conjugate that is the same as the fourth equation, and the second equation has a complex conjugate that is the same as the third equation. Therefore, we can disregard the last equation and the second equation, respectively. Thus, we can obtain a reduced set of equations consisting of the first and third equations.

$$\begin{aligned} a\alpha - \bar{b}\beta &= \alpha a - \bar{\beta}b \\ b\alpha + \bar{a}\beta &= \beta a + \bar{\alpha}b \end{aligned}$$

The first equation gives:

$$a\alpha - \bar{b}\beta = \alpha a - \bar{\beta}b \iff a\alpha - \alpha a = \bar{\beta}b - \bar{b}\beta \iff \bar{\beta}b - \bar{b}\beta = 0$$

Now add to these the conditions $|a|^2 + |b|^2 = 1$, $|\alpha|^2 + |\beta|^2 = 1$. Then write each complex number into its Cartesian form. The set X will be described in terms of equations with real coefficients. These are the equations we used in our codes to generate point clouds.

Theorem 7.1. *The homology groups of $SU(2)$ can be computed from the topology of the 3-sphere S^3*

Proof. In fact, $SU(2)$ is diffeomorphic to the 3-sphere S^3 , so their homology groups are isomorphic. To see the diffeomorphism between $SU(2)$ and S^3 , we can use the fact that any element $g \in SU(2)$ can be written as a 2×2 unitary matrix with determinant 1, i.e., $g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. We can identify such a matrix with a point $(a, b) \in \mathbb{C}^2$ satisfying $|a|^2 + |b|^2 = 1$. This gives a bijection between $SU(2)$ and the unit 3-sphere $S^3 \subseteq \mathbb{C}^2$. Moreover, this bijection is a diffeomorphism, which means that the smooth structure on $SU(2)$ is the same as the smooth structure on S^3 .

Since S^3 is a simply-connected space, its homology groups are particularly easy to compute. In fact, the homology groups of S^3 are as follows:

- $H_0(S^3) \cong \mathbb{Z}$, the group of connected components of S^3 .
- $H_1(S^3) \cong 0$, the group of loops in S^3 $H_2(S^3) \cong 0$, the group of 2-dimensional surfaces in S^3 .
- $H_3(S^3) \cong \mathbb{Z}$, the group of 3-dimensional volumes in S^3 Since $SU(2)$ is diffeomorphic to S^3 , it follows that the homology groups of $SU(2)$ are isomorphic to those of S^3 . This means that $H_0(SU(2)) \cong \mathbb{Z}$ and $H_k(SU(2)) \cong 0$ for $k = 1, 2$, while $H_3(SU(2)) \cong \mathbb{Z}$.

□

8. n -SIMPLEX AND SIMPLICIAL COMPLEX

Definition 8.1. A k -dimensional **simplex** σ^k is defined as the convex hull of $(k + 1)$ affinely independent points $v_0, v_1, \dots, v_k \in \mathbb{R}^n$. In other words, a k -simplex is the smallest convex set in \mathbb{R}^n that contains the points v_0, v_1, \dots, v_k and that is not contained in any affine subspace of dimension less than k .

More formally, we can define a k -simplex as follows:

Let $V = \{v_0, v_1, \dots, v_k\}$ be a set of $(k + 1)$ points in \mathbb{R}^n that are affinely independent, i.e., not contained in any $(k - 1)$ -dimensional affine subspace. The k -simplex σ^k with vertices v_0, v_1, \dots, v_k is the set of all convex combinations of these vertices, i.e.,

$$\sigma^k = \left\{ \sum_{i=0}^k \lambda_i v_i \mid \lambda_i \geq 0, \sum_{i=0}^k \lambda_i = 1 \right\}$$

Geometrically, a k -simplex can be visualized as the $(k+1)$ -dimensional analogue of a triangle, tetrahedron, or any other regular polygon in lower dimensions.

Example 8.1. • A 0-simplex is a single point.

- A 1-simplex is a line segment connecting two points.
- A 2-simplex is a triangle with three vertices and three edges.
- A 3-simplex is a tetrahedron with four vertices, six edges, and four faces.
- A 4-simplex is a five-dimensional polytope with five vertices, ten edges, ten faces, and five cells.
- More generally, an n -simplex is the convex hull of $n + 1$ affinely independent points in Euclidean space. So, for example, a 5-simplex is the convex hull of 6 points in 5-dimensional space.

Definition 8.2. A **simplicial complex** is a topological space constructed from a collection of simplices that satisfies the following conditions:

- Closure under faces: If a simplex is in the collection, then all of its faces are also in the collection.
- Intersection property: The intersection of any two simplices in the collection is a face (or empty).

Formally, let \mathcal{K} be a collection of simplices in some Euclidean space, \mathbb{R}^n . Then \mathcal{K} is a simplicial complex if it satisfies the following two conditions:

- For every simplex σ in \mathcal{K} , every face of σ is also in \mathcal{K} .
- For any two simplices σ_1, σ_2 in \mathcal{K} , their intersection $\sigma_1 \cap \sigma_2$ is either empty or a face of both σ_1 and σ_2 .

The simplices in the collection are typically required to be finite and of finite dimension. The **dimension** of a simplicial complex is the maximum dimension of any simplex in the collection.

Example 8.2. *Here are some examples of simplicial complexes:*

- *A 0-dimensional simplicial complex is simply a collection of points.*
- *A discrete simplicial complex: This is a collection of points, where each point is a 0-simplex in the complex. There are no higher-dimensional simplices.*
- *A line segment: This is a 1-dimensional simplicial complex, consisting of two 0-simplices (the endpoints) and a single 1-simplex (the line segment connecting the endpoints).*
- *A triangle: This is a 2-dimensional simplicial complex, consisting of three 0-simplices (the vertices), three 1-simplices (the edges connecting the vertices), and a single 2-simplex (the triangle formed by the three vertices).*
- *A tetrahedron: This is a 3-dimensional simplicial complex, consisting of four 0-simplices (the vertices), six 1-simplices (the edges connecting the vertices), four 2-simplices (the faces of the tetrahedron), and a single 3-simplex (the tetrahedron itself).*

Example 8.3. *A torus can be represented as a simplicial complex by gluing together the edges of a square in a particular way. We start with a square, which can be thought of as a 2-simplex, and then subdivide it into two triangles, which are each 2-simplices. The resulting simplicial complex has:*

- *Four vertices, each corresponding to a corner of the square.*
- *Four 1-simplices, each corresponding to an edge of the square.*
- *Two 2-simplices, each corresponding to one of the triangles formed by the subdivision of the original square.*

To obtain the torus, we identify the top edge of the square with the bottom edge, and the left edge with the right edge, but with a "twist" so that the identification is not trivial. This can be visualized as taking the square, bending it into a cylinder, and then gluing the two ends of the cylinder together with a twist.

The resulting simplicial complex has a nontrivial first homology group, which corresponds to the "hole" in the torus. Specifically, there is a cycle (a closed loop) in the complex that cannot be continuously deformed to a point, corresponding to a nontrivial element of the first homology group.

Definition 8.3. An n -chain is a formal linear combination of n -simplices in a simplicial complex. It is written as

$$c = \sum_i a_i \sigma_i,$$

where σ_i are n -simplices in the simplicial complex and a_i are coefficients from some field (such as the real or complex numbers). The sum is finite and nonzero for only finitely many i .

Geometrically, an n -chain can be thought of as a collection of n -dimensional objects (such as triangles, tetrahedra, or higher-dimensional simplices) that are weighted by some coefficients. For example, a 2-chain in a simplicial complex could be a weighted sum of triangles, where the weights correspond to the areas of the triangles.

9. HOMOLOGY

Homology is a way of assigning algebraic objects (groups) to topological spaces that capture their topological features. Specifically, the homology groups of a space measure the number and size of the "holes" of various dimensions in the space.

Definition 9.1. A cycle is a formal linear combination of simplices in a simplicial complex whose boundary is zero. More specifically, an n -cycle in a simplicial complex X is an n -chain c in X such that the boundary of c is zero, i.e.,

$$\partial_n c = 0,$$

where ∂_n is the boundary operator that maps an n -chain to an $(n-1)$ -chain. To be more specific, the boundary operator ∂_n is a linear map that maps an n -chain in a simplicial complex to an $(n-1)$ -chain, and is defined as the formal sum of the $(n-1)$ -dimensional faces of the n -simplex, each with a sign depending on its orientation.

Intuitively, a cycle can be thought of as a closed loop or surface that does not have any boundary or edge. For example, a circle in a 2-dimensional simplicial complex can be represented as a 1-cycle.

The set of all cycles in a simplicial complex X , is denoted by Z_n and defined as the kernel of the boundary operator ∂_n . That is,

$$Z_n = \ker(\partial_n).$$

Definition 9.2. A **boundary** is a formal linear combination of $(n-1)$ -simplices in a simplicial complex X that lie on the boundary of an n -simplex in X .

More formally, the n -th boundary group $B_n(X)$ is the subgroup of the n -th chain group $C_n(X)$ generated by the boundaries of all $(n + 1)$ -simplices in X :

$$B_n(X) = \{\partial(\sigma) : \sigma \text{ is an } (n + 1)\text{-simplex in } X\}$$

where σ is the boundary operator.

Intuitively, a boundary can be thought of as the edge or boundary of a simplex or surface. For example, in a 2-dimensional simplicial complex, the boundary of a triangle is a cycle consisting of its three edges.

Definition 9.3. The n th homology group of a topological space X , denoted $H_n(X)$, is defined as the quotient group of the n th cycle group, $Z_n(X)$, modulo the n th boundary group, $B_n(X)$:

$$H_n(X) = Z_n(X)/B_n(X)$$

where a cycle is a chain (a formal linear combination of simplices) that is the boundary of another chain, and a boundary is a chain that lies entirely in the boundary of the space X .

In the context of topological data analysis (TDA), homology provides a powerful tool for analyzing the shape of data sets, which can be thought of as sampled from an unknown topological space. By computing the homology groups of the simplicial complexes built from the data, we can extract topological features such as connected components, loops, voids, and higher-dimensional structures. This allows us to gain insight into the shape and structure of the underlying space, which can be useful in a wide range of applications, including computer vision, machine learning, and materials science.

10. LIMITATION AND CHALLENGES

10.1. **Limitations.** The following are limitations of our current approach:

- Our current approach is restricted to computing features of dimensions 0 and 1, while there may exist **higher dimensional features** up to dimension 4 in the data. The inability to compute these features may lead to a limited understanding of the underlying topological structure.
- Our method of sampling point cloud may introduce bias in the form of an increased number of points in the form of (A, I) , where I denotes the identity matrix. This may have an impact on the accuracy of our results.

10.2. Challenges.

- The computational cost associated with creating simplices from point clouds in higher dimensions is prohibitively expensive, both in terms of time and memory complexity, posing a significant challenge for analyzing point clouds with a large number of dimensions.
- The current sample size used in this study may not be sufficient to derive meaningful results for higher dimensional features. Obtaining additional sample points could potentially address this issue.

11. FUTURE DIRECTIONS

As our work has shown promising results in studying the topological properties of the space of commuting pairs of $SU(2)$, there are several directions for future research that could build upon our findings and expand their scope. These include:

- Developing a **new algorithm** that can sample random points more uniformly, thereby resulting in a more accurate representation of the data.
- Creating a more **efficient algorithm** that can handle higher dimensional and more complex data, while minimizing both the time and memory complexity.
- Generalizing our method to **other Lie groups** to gain insights into the topological properties of these groups and their representations.
- Discovering a **different metric** that can reduce the computational cost in higher dimensions while still preserving important topological features.

12. CONCLUSION

In conclusion, the main objective of this project was to investigate the potential of TDA in analyzing the topological properties of point clouds obtained from the set of commuting pairs in the group $SU(2)$. The use of persistent homology and persistent homology barcodes facilitated the extraction of topological features from the point cloud and enabled us to obtain a deeper understanding of its underlying structure. Our findings revealed that the space X consists of a single connected component and contains no topologically non-trivial loops, which corroborates the results of earlier studies. However, the exponential computational complexity associated with computing persistence diagrams limited our ability to derive conclusive results from higher dimensions.

The method is limited to computing features up to dimension 1, and is affected by the non-uniform sampling of the point cloud. To address these limitations, we suggest future directions such as developing a new algorithm for more uniform point cloud sampling, developing a more efficient algorithm for handling higher dimensional and more complex data, generalizing the method to other lie groups, and discovering different metrics that could reduce the computational cost in higher dimensions. Overall, the method presented in this study provides insights into the topological properties of the space of commuting pairs of $SU(2)$ matrices and lays the foundation for future studies on the topological properties of other spaces.

REFERENCES

- [1] Thomas Baird. “Cohomology of the space of commuting n-tuples in a compact Lie group”. In: *arXiv preprint math/0610761* (2006).
- [2] Thomas Baird and Andrew Baker. “Commuting elements and spaces of homomorphisms”. In: *arXiv preprint math/0603197* (2006).
- [3] Gunnar Carlsson. “Topology and data”. In: *Bulletin of the American Mathematical Society* 45.2 (2008), pp. 255–308.
- [4] David L Donoho, Jared Tanner, and Martin A Gavish. “A klein-bottle-based dictionary for texture representation”. In: *arXiv preprint arXiv:1107.1768* (2011).
- [5] Robert Ghrist. “Barcodes: The persistent topology of data”. In: *Bulletin of the American Mathematical Society* 45.1 (2008), pp. 61–75.
- [6] Jose A Perea and John Harer. “Sliding windows and persistence: An application of topological methods to signal analysis”. In: *Foundations of Computational Mathematics* 15.3 (2015), pp. 799–838.