Topological Data Analysis of Commuting Pairs of $\mathrm{SU}(2)$

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Introduction

Topology is a branch of mathematics that studies the properties of objects that remain unchanged when the object is stretched, bent, or twisted. In recent years, topological methods have become increasingly popular in data analysis, as they provide a powerful tool for extracting meaningful information from complex datasets.

One key concept in topology is **homology**, which measures the number of holes in an object of a given dimension. Homology provides a way to distinguish different shapes and structures and has been used to analyze a wide range of datasets, from images and videos to gene expression data and networks.

Topological Data Analysis (TDA) is a field that uses algebraic topology to analyze complex data sets. One of the main techniques in TDA is **Persistence Homology**, which assigns a measure of persistence to topological features in a data set. In this project, we apply Persistence Homology to a data set of points of commuting pairs of matrices in the Lie group $\mathrm{SU}(2)$. We used the Ripser software to compute the persistence homology of our data set. **Ripser** is a C++ program that computes persistence homology using the **Vietoris-Rips filtration**.

Our method uses a combination of topological and algebraic techniques to construct a simplicial complex from a point cloud and then computes the persistent homology of this complex to identify the topological features of the space.

Our results show that there is a connected component in the space of commuting pairs of matrices in $\mathrm{SU}(2)$, but no higher-dimensional features such as loops. We compare our results with previous work in the field and identify several limitations and challenges that need to be addressed in future research.

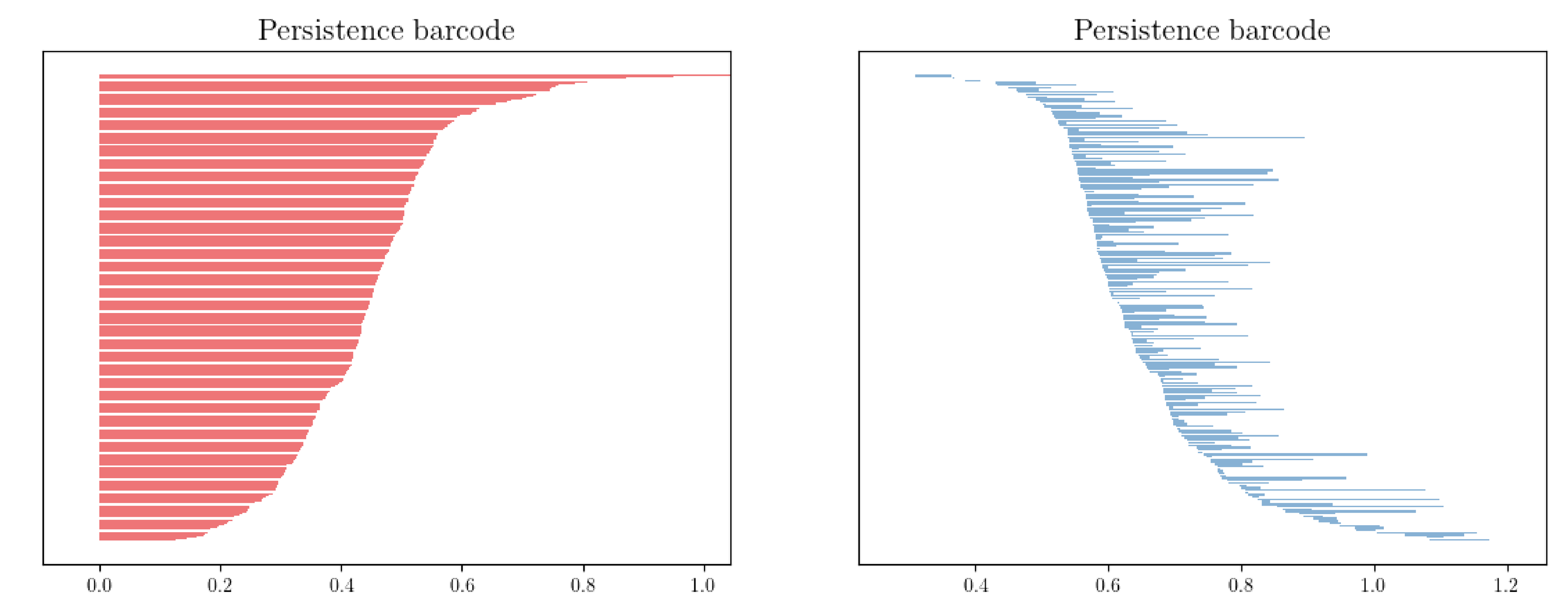
Methodology

This project employed TDA to analyze the topological properties of point clouds obtained from the set of commutating pairs in $\mathrm{SU}(2)$. The following steps were followed:

- **Data collection**: The data set used in this project was obtained from the set of commutating pairs in $\mathrm{SU}(2)$. Specifically, for each pair of commuting elements in $\mathrm{SU}(2)$, a point was obtained in the four-dimensional Euclidean space $\mathbb{R}^8$. The data set thus obtained consisted of 1000 points.
- **Persistent homology**: Persistent homology was employed to extract topological features from the preprocessed data set. In particular, we utilized the Rips filtration, which constructs a simplicial complex from the point cloud by connecting points that are within a certain distance of each other. The distance was varied over a range of values, and the persistence diagram was computed for each value. The persistence diagram provides a visual representation of the birth and death of topological features (i.e., connected components, loops, higher dimensional holes) as the distance parameter is increased.
- **Analysis of persistent homology**: The persistence diagram was analyzed to determine the topological features present in the data set. In particular, we examined the zeroth and first homology groups to determine the number of connected components and loops in the data set, respectively.
- **Computation of persistent homology barcodes**: In addition to the persistence diagram, persistent homology barcodes were computed to obtain a more detailed characterization of the topological features present in the data set. The barcode provides information about the birth and death times of topological features as well as their dimensions.

Overall, the methodology employed in this project provides a comprehensive approach to analyzing the topological properties of point clouds using TDA. The combination of preprocessing, persistent homology, and statistical analysis ensures that meaningful insights can be obtained from the data set.

Results



(a) Barcode of dimension 0 (b) Barcode of dimension 1

Figure 1. Barcode of dimension 0 and 1 of the data sampled from the space of commuting pairs in $\mathrm{SU}(2)$

Interpretation

The left figure's barcode indicates that the point cloud has a single 0-dimensional feature, specifically a connected component that persists until the maximum edge length of 10000. This feature is represented by the topmost line on the barcode. Conversely, the right figure's barcode does not contain any lines that are long enough to be considered significant. Consequently, we can infer that there are no topologically non-trivial loops in our point cloud.

Comparison with existing result

Our findings align with the results derived by Adem and Cohen, which are expressed as:

$$H^i(\mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z},\mathrm{SU}(2)),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if} \quad i=0 \\ 0 & \text{if} \quad i=1 \end{cases}$$

This equation indicates that the space of commuting pairs of $\mathrm{SU}(2)$ has a connected component and no loops, which corresponds to our TDA results.

Limitation and Challenges

Limitations

- Our current approach is restricted to computing features of dimensions 0 and 1, while there may exist **higher dimensional features** up to dimension 4 in the data. The inability to compute these features may lead to a limited understanding of the underlying topological structure.
- Our method of sampling point cloud may introduce bias in the form of an increased number of points in the form of (A, I), where I denotes the identity matrix. This may have an impact on the accuracy of our results.

Challenges

- The computational cost associated with creating simplices from point clouds in higher dimensions is prohibitively expensive, both in terms of time and memory complexity, posing a significant challenge for analyzing point clouds with a large number of dimensions.
- The current sample size used in this study may not be sufficient to derive meaningful results for higher dimensional features. Obtaining additional sample points could potentially address this issue.

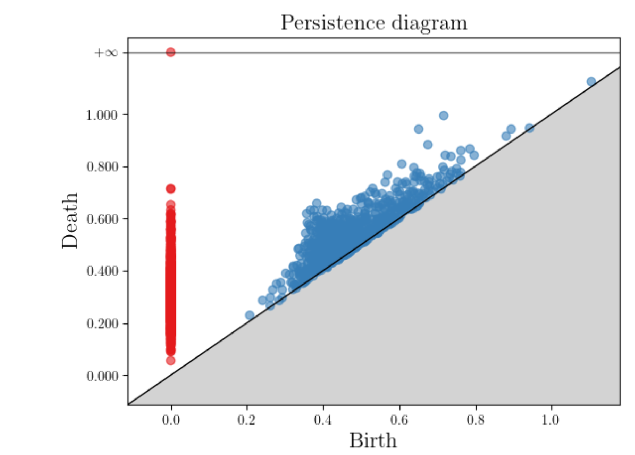


Figure 2. Persistent diagram

Future Directions

As our work has shown promising results in studying the topological properties of the space of commuting pairs of $\mathrm{SU}(2)$, there are several directions for future research that could build upon our findings and expand their scope. These include:

- Developing a **new algorithm** that can sample random points more uniformly, thereby resulting in a more accurate representation of the data.
- Creating a more **efficient algorithm** that can handle higher dimensional and more complex data, while minimizing both the time and memory complexity.
- Generalizing our method to **other Lie groups** to gain insights into the topological properties of these groups and their representations.
- Discovering a **different metric** that can reduce the computational cost in higher dimensions while still preserving important topological features.

Conclusion

In conclusion, the main objective of this project was to investigate the potential of TDA in analyzing the topological properties of point clouds obtained from the set of commuting pairs in the group $\mathrm{SU}(2)$. The use of persistent homology and persistent homology barcodes facilitated the extraction of topological features from the point cloud and enabled us to obtain a deeper understanding of its underlying structure. Our findings revealed that the space X consists of a single connected component and contains no loops, which corroborates the results of earlier studies. However, the exponential computational complexity associated with computing persistence diagrams limited our ability to derive conclusive results from higher dimensions. The method is limited to computing features up to dimension 1, and is affected by the non-uniform sampling of the point cloud. To address these limitations, we suggest future directions such as developing a new algorithm for more uniform point cloud sampling, developing a more efficient algorithm for handling higher dimensional and more complex data, generalizing the method to other lie groups, and discovering different metrics that could reduce the computational cost in higher dimensions. Overall, the method presented in this study provides insights into the topological properties of the space of commuting pairs of $\mathrm{SU}(2)$ matrices and lays the foundation for future studies on the topological properties of other spaces.

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